## A TEST FOR SECOND-ORDER STOCHASTIC DOMINANCE

## By S. Rao Jammalamadaka

Department of Statistics and Applied Probability
University of California, Santa Barbara

and

#### **Eun-Soo Lee**

Statistical Analysis Division, National Statistical Office
Seoul, Korea.

#### SUMMARY

A cumulative distribution function F is said to stochastically dominate another distribution function G in the second-order sense if  $\int_0^x F(t) dt \le \int_0^x G(t) dt$ , for all x with strict inequality for some x. It is desired to test the null hypothesis  $H_0$ : F=G versus  $H_1$ : F stochastically dominates G in the second-order sense. The aim of this paper is to develop a test, which is sensitive to this one-sided alternative. A test based on the empirical distribution functions is proposed and its asymptotic null distribution derived, making this test procedure useful in large samples.

Keywords: Second-order stochastic dominance; Brownian Bridge; Continuous mapping theorem; Jackknife variance estimator.

1991 AMS Subject Classification: Primary 62E20, 62G10; Secondary 62G20, 62G30.

#### 1. Introduction

In economics, finance as well as in statistics, stochastic dominance theory has become an important tool in the analysis of choice under uncertainty.

Given two random prospects X and Y with Cumulative Distribution Functions (CDF's) F and G respectively, both defined on the whole real line, we would like to define an order of preference between F and G. Assume every

individual has von Neumann-Morgenstern utility function u(x) and ranks random prospects by his/her expected utility,  $E_F[u(X)]$ . Let  $U_i$  for i=1,2 denote utility function classes, where

(i) 
$$U_1 = \{ u: \frac{\partial}{\partial x} u(x) \ge 0 \ \forall x \}$$

(ii) 
$$U_2 = \{ u: \frac{\partial}{\partial x} u(x) \ge 0, \frac{\partial^2}{\partial x^2} u(x) \le 0 \ \forall \ x \}.$$

 $\rm U_1$  is an extremely large class containing practically all acceptable utility functions. Utility functions in  $\rm U_2$  exhibit decreasing marginal utility, and individuals having these utility functions are called "risk-averse". Although some individuals may be risk-seekers, firms, governments and most people are generally more conservative, and we would expect the utility function they employ, to belong to the class  $\rm U_2$ .

Two types of Stochastic Dominance (hereafter SD) are commonly used: First-order SD (FSD) and Second-order SD (SSD).

- (i) F stochastically dominates G in the first order if and only if  $E_F[u(X)] \ge E_G[u(Y)]$  for all u in U<sub>1</sub>.
- (ii) F stochastically dominates G in the second order if and only if  $E_F[u(X)] \geq E_G[u(Y)] \ \ \text{for all } u \ \text{in } U_2.$

FSD and SSD have convenient characterizations in terms of the distribution functions of the prospects given by the following result (cf. Hadar and Russell (1969) and Hanoch and Levy (1969)):

(i)  $F(x) \le G(x)$ , for all x with strict inequality for some x iff  $E_F[u(X)] \ge E_G[u(Y)] \text{ for all } u \text{ in } U_T.$ 

(ii)  $\int_0^x F(t) dt \le \int_0^x G(t) dt$ , for all x with strict inequality for some x iff  $E_F[u(X)] \ge E_G[u(Y)]$  for all u in  $U_2$ .

This result is the key to stochastic dominance. Suppose X dominates Y in the sense of SSD. Then no risk-averse individual would ever choose Y, because the expected utility of X is greater than the expected utility of Y for all risk-averse utility functions. Clearly in order that a well-defined answer exist to the question whether

$$E_{F}[u(X)] \ge E_{G}[u(Y)]$$
 for all u in  $U_{i}$ , i=1, 2

for any pair of distribution functions, a summary measure is needed. One can then at least order the measures for the two random prospects to decide which one provides larger utility. So the problem of comparing and choosing between two random prospects X and Y, is equivalent to testing for FSD in case of utility functions belonging to the class  $U_1$  and to testing for SSD in case of utility functions belonging to the class  $U_2$ . If F dominates G in the first order then F also dominates G in the second order but the converse is not always true. Our interest is focused on the SSD.

Section 2 presents a test statistic,  $T_N$ , based on the empirical CDF. We obtain the asymptotic distribution of this statistic  $T_N$  under the null hypothesis. In section 3, a method of estimating this asymptotic variance is provided so that the test can be put to practical use.

# 2. Testing For SSD Based on the Empirical CDF

There is considerable literature and adequate statistical theory for testing for FSD. Several nonparametric procedures have also been developed (see e.g. Whitmore and Findlay (1978), McFadden (1989) and the references contained there). For the case of SSD, however, there do not appear to be any

satisfactory tests. Deshpande and Singh (1985) developed a one sample test for testing

$$H_0: F = F_0$$

versus

 $H_1$ : F stochastically dominates  $F_0$  in the second order sense,

where  $F_0$  is a fully specified and known CDF. They define

$$D_{F,F_0} = \int_{-\infty}^{\infty} \int_{-\infty}^{x} (F(t) - F_0(t)) dt dF_0(x)$$

and base their test on the sample version of D<sub>F,F<sub>0</sub></sub>, given by

$$D_n = \int_{-\infty}^{\infty} \int_{-\infty}^{x} (F_n(t) - F_0(t)) dt dF_0(x),$$

where  $F_n(x)$  is the empirical CDF. They showed that the standardized version of  $D_n$  is asymptotically normally distributed. McFadden (1989) proposes a test for testing SSD

$$H_0: \int_0^w F(y) dy \le \int_0^w G(y) dy$$
 for all w in [0, 1]

versus

$$H_1: \int_0^w F(y) dy > \int_0^w G(y) dy$$
 for some w in [0, 1].

His test statistic is the sample analog

$$S_n^* = \text{Max } S_n(w), w \in [0, 1], \text{ where } S_n(w) = n^{1/2} \int_0^w \{G_n(y) - F_n(y)\} dy.$$

However, the asymptotic distribution of  $S_n^*$  under  $H_0$  is not fully known. Furthermore, he assumes equal sample sizes from the two populations. See

also Kaur et al (1994) for a more satisfactory approach analogous to ours.

Our interest is focused on developing test procedures for the two sample SSD problem which do not rely on parametric assumption. Let  $X_1, ..., X_n$  and  $Y_1, ..., Y_m$  denote random samples from continuous CDFs F and G, respectively. Here we will consider the problem of testing

$$H_0: F = G$$

versus

H<sub>1</sub>: F dominates G by SSD

on the basis of the two observed samples  $X_1, ..., X_n$  and  $Y_1, ..., Y_m$ . If  $F_n(t)$  and  $G_m(t)$  denote the empirical distribution functions of these two samples respectively, then a test procedure which rejects  $H_0$  if  $\int_{-\infty}^x \{G_m(t) - F_n(t)\} dt$  is large, has some intuitive appeal but this depends on the value x. Let w(x) be some pre-assigned non-negative weight function to be chosen by the statistician so as to weight the deviations according to the importance attached to various portions of the distribution function or taken to be a constant if there is no preferential weighting which makes sense. We now introduce the statistic

$$T_{N} = \sqrt{\frac{nm}{N}} \int_{-\infty}^{\infty} w(x) \left\{ \int_{-\infty}^{x} (G_{m}(t) - F_{n}(t)) dt \right\} dH_{N}(x),$$

where  $H_N(x) = (n/N) F_n(x) + (m/N) G_m(x)$ , x in  $\Re$ , and N=n+m. We assume that there exists a limit  $\rho$ ,  $0 < \rho < 1$ , such that

$$n/N \rightarrow \rho$$
 as  $N \rightarrow \infty$ .

The normalizing factor  $(nm/N)^{1/2}$  is included to produce proper non-degenerate asymptotic distributions under  $H_0$ . Define

$$D_{N}(x) = w(x) \int_{-\infty}^{x} (G_{m}(t) - F_{n}(t)) dt.$$

For  $x \ge 0$ ,  $D_N(x)$  is a weighted measure of the deviation from  $H_0$  toward  $H_1$ , and  $T_N$  is an average value of this deviation. It is important to note that stochastic ordering corresponds to one-sided alternative and certainly any test procedure should be sensitive to this alternative.

Using the probability integral transformation  $U_i = F(X_i)$ , and  $V_j = F(Y_j)$ , define

$$B_n(u) = n^{-1/2} \sum_{i=1}^n (I(U_i \le u) - u), \text{ and } B_m(u) = m^{-1/2} \sum_{j=1}^m (I(V_j \le u) - u).$$

and

$$B_{N}(u) = \sqrt{\frac{n}{N}} B_{m}(u) - \sqrt{\frac{m}{N}} B_{n}(u), \quad 0 \le u \le 1.$$

Then we have

$$T_N = \int_0^1 w(s) \{ \int_0^s (B_N(u)/f(F^{-1}(u))) du \} ds$$

Note that each of  $B_n$  and  $B_m$  converges weakly to a Brownian bridge  $B_{01}$  and  $B_{02}$  respectively. Also since  $B_n$  and  $B_m$  are independent, so are  $B_{01}$  and  $B_{02}$ . Therefore,  $B_N$  being a linear function of  $B_n$  and  $B_m$ , we have the result that

$$B_N \xrightarrow{D} \sqrt{\rho} B_{02} - \sqrt{1-\rho} B_{01} \equiv B_0$$
, say

as  $N \to \infty$ . It follows that if  $B_N = \{ B_N(u) : 0 \le u \le 1 \}$ , then

$$B_N \xrightarrow{D} B_0$$
 as  $N \to \infty$ 

**Theorem 2.1.** Under  $H_0$ , as  $N \to \infty$ ,  $T_N$  has a limiting  $N(0, \sigma^2)$  distribution, where  $\sigma^2$  is given by

$$\sigma^{2} = \int_{0}^{1} \int_{0}^{1} w(s)w(t) \int_{0}^{t} \int_{0}^{s} \frac{1}{f(F^{-1}(u))} \frac{1}{f(F^{-1}(v))} [\min(u,v)-uv] dudv dsdt \quad (1)$$

For the proof of Theorem 2.1 we require the following Lemmas.

Lemma 2.2. As  $N \rightarrow \infty$ ,

$$\int_0^s (B_N(u)/f(F^{-1}(u))) du \xrightarrow{D} \int_0^s (B_0(u)/f(F^{-1}(u))) du, \quad 0 \le s \le 1.$$

**Proof.** To prove that

$$\int_0^s (B_N(u)/f(F^{-1}(u))) du \xrightarrow{D} \int_0^s (B_0(u)/f(F^{-1}(u))) du,$$

for every fixed s,  $0 \le s \le 1$ , it is sufficient to show that the mapping

$$h(B_N(u)) = \int \{B_N(u)/f(F^{-1}(u))\}du$$

is (a.e.) continuous in the Skorohod metric (say  $d_s$ ), so that the result follows by the continuous mapping theorem (cf. Billingsley(1968) p.30). First, because of the assumptions on f and F, f( $F^{-1}(u)$ ) is continuous for  $0 \le u \le 1$ . It is then easy to show that  $g_n(u) = B_n(u)/f(F^{-1}(u))$ , when regarded as a function of  $B_n(u)$ , is continuous in  $d_s$  so that

$$g_n(u) \xrightarrow{D} g(u) = B_0(u)/f(F^{-1}(u)).$$

The result

nd

ilt

$$\int_0^s (B_N(u)/f(F^{-1}(u))) du \xrightarrow{D} \int_0^s (B_0(u)/f(F^{-1}(u))) du, \quad 0 \le s \le 1$$

will then follow if we show that  $\int_0^s g(u) du$ , as a functional of  $g(\bullet)$ , is continuous in  $d_s$ .

Now for any sequence of functions  $g_k$  converging to g in  $d_s$  there exist functions  $\lambda_k$  such that  $\lim_{k\to\infty} g_k(\lambda_k(u)) = g(u)$  uniformly in u and  $\lim_{k\to\infty} \lambda_k(u) = u$  uniformly in u (Skorohod topology: see Billingsley(1968) for details). Since every element of D is bounded and has at most a countable number of discontinuities, it is Riemann integrable.

Let  $P_h$ :  $0 < u_1 < ... < u_h < s$  and  $P_h'$ :  $0 < \lambda_k(u_1) < ... < \lambda_k(u_h) < s$  be two sequences of partions such that  $P_h$  is for  $\int_0^s g(u) du$  and  $P_h'$  is for  $\int_0^s g_k(u) du$ . As  $k \to \infty$  the upper and lower sums for the latter integral converge to those for the former. Therefore it follows that

$$\int_0^s g_k(u) du \to \int_0^s g(u) du, 0 \le s \le 1$$

and that  $\int_0^s g(u) du$  is continuous in  $d_s$ .

Lemma 2.3. As  $N \to \infty$ ,

$$\int_0^1 w(s) \{ \int_0^s (B_N(u)/f(F^{-1}(u))) du \} ds$$

$$\xrightarrow{D} \int_0^1 w(s) \{ \int_0^s (B_0(u)/f(F^{-1}(u))) du \} ds$$

for any pre-assigned nonnegative function, w(s),  $0 \le s \le 1$ .

Proof. By Lemma 2.2,

$$\int_0^s (B_N(u)/f(F^{-1}(u))) du \xrightarrow{D} \int_0^s (B_0(u)/f(F^{-1}(u))) du$$

as  $N \to \infty$ . Let  $D_T$  be the set of discontinuities of  $\int_0^s (B_0(u)/f(F^{-1}(u))) du$ . Since X and w are independent with X being continuous,  $D_T \cap D_w = \emptyset$  and  $P\{(D_T \cap D_w) \neq \emptyset\} = 0$ . Thus, multiplication on  $D \times D$ , defined by

$$w(\{\int_0^s (B_0(u)/f(\mathit{F}^{-1}(u))) \, du\}s) = w(s) \, \{\int_0^s (B_0(u)/f(\mathit{F}^{-1}(u))) \, du\}, \, 0 \leq s \leq 1,$$

is measurable on D×D and continuous at those  $\int_0^s (B_0(u)/f(F^{-1}(u))) du$ , w for which  $D_T \cap D_w = \emptyset$  (cf. Whitt (1980) p.79). Thus

$$w(s)\{\int_0^s (B_N(u)/f(F^{-1}(u)))du\} \xrightarrow{D} w(s)\{\int_0^s (B_0(u)/f(F^{-1}(u))) du\}$$
 and shence

$$\int_{0}^{1} w(s) \{ \int_{0}^{s} (B_{N}(u)/f(F^{-1}(u)))du \} ds$$

$$\xrightarrow{D} \int_{0}^{1} w(s) \{ \int_{0}^{s} (B_{0}(u)/f(F^{-1}(u))) du \} ds$$

will then follow from the continuity of the integral operation in conjunction with the continuous mapping theorem.

**Lemma 2.4.** Let w(s),  $0 \le s \le 1$ , be a deterministic piecewise continuous function. Then the stochastic integral

$$\int_0^1 w(s) \{ \int_0^s (B_0(u)/f(F^{-1}(u))) du \} ds$$

exists in the sense of convergence in mean square and has a N(0,  $\sigma^2$ ) distribution, where  $\sigma^2$  is given by (1).

Proof. As a first step, define the stochastic integral

$$\int_0^s (B_0(u)/f(F^{-1}(u))) du,$$

where  $\{B_0(u); 0 \le u \le 1\}$  is a Brownian bridge.  $B_0(u)$  is a Gaussian process whose mean value function and covariance kernel are given by

$$EB_0(u) = 0$$
, and  $Cov(B_0(u), B_0(v)) = min(u, v) - uv$ .

These are continuous functions of u and v as is  $1/f(F^{-1}(u))$ . Let

$$0 = u_{p,0} < u_{p,1} < ... < u_{p,p} = s$$

such that

$$\lim_{p\to\infty}\max_{1\leq k\leq p}\left(\,u_{p,k}\,\,-\,\,u_{p,k-1}\,\right)=0.$$

We form the "Riemann sums"

$$S_{p} = \sum_{k=1}^{p} \frac{B_{0}(u_{p,k}^{*})}{f(F^{-1}(u_{p,k}^{*}))} (u_{p,k} - u_{p,k-1}),$$

while

$$S_p S_q = \sum_{k=1}^p \sum_{j=1}^q \frac{B_0(u_{p,k}^*) B_0(v_{q,j}^*)}{f(F^{-1}(u_{p,k}^*)) f(F^{-1}(v_{q,j}^*))} (u_{p,k} - u_{p,k-1}) (v_{q,j} - v_{q,j-1}),$$

where  $u_{p,k-1} \leq u_{p,k}^* \leq u_{p,k}$  and  $v_{q,j-1} \leq v_{q,j}^* \leq v_{q,j}$  . Then

$$E(S_p S_q) = \sum \sum \frac{\min(u_{p,k}^*, v_{q,j}^*) - u_{p,k}^* v_{q,j}^*}{f(F^{-1}(u_{p,k}^*)) f(F^{-1}(v_{q,j}^*))} (u_{p,k} - u_{p,k-1}) (v_{q,j} - v_{q,j-1}).$$

and we obtain

$$\lim_{\rho,q\to\infty} E(S_{\rho}S_{q}) = \int_{0}^{t} \int_{0}^{s} \frac{\min(u,v) - uv}{f(F^{-1}(u))f(F^{-1}(v))} dudv.$$

This limit is the same for all sequence of subdivisions and all choices of the intermediate points. Now appealing to Loeve's Theorem 6.1.5 (1963, p.469), we see that the "Riemann sum" Sp converges in mean square. It follows from the fact that the Brownian bridge is a Gaussian process that

$$\int_0^s (B_0(u)/f(F^{-1}(u))) du$$

is also a Gaussian. Its distribution is characterized by the mean value and covariance function, which are given by

$$E(\int_0^s (B_0(u)/f(F^{-1}(u))) du) = \int_0^s EB_0(u) du = 0$$
, for all u

and

$$k(s, t) = Cov[\int_0^s (B_0(u)/f(F^{-1}(u))) du, \int_0^t (B_0(u)/f(F^{-1}(u))) du]$$

$$= \int_0^t \int_0^s \frac{1}{f(F^{-1}(u))} \frac{1}{f(F^{-1}(v))} [\min(u, v) - uv] du dv$$
 (2)

In a similar way, the integral

$$\int_0^1 \int_0^1 w(s)w(t)k(s, t) ds dt$$

exists since w(s) is a piecewise continuous function on I=[0, 1] and also k(s, t) is continuous on  $I^2$ . Thus the integral

$$\int_0^1 w(s) \left\{ \int_0^s (B_0(u)/f(F^{-1}(u))) du \right\} ds$$

is well defined as a limit in mean square of the usual approximating sums. It

can be shown that this integral defines a random variable. Since the approximating sum of this integral is the sum of independently and normally distributed random variables, we see that

$$\int_0^1 w(s) \{ \int_0^s (B_0(u)/f(F^{-1}(u))) du \} ds$$

is also normally distributed. Moreover it is easily seen that

$$E[\int_0^1 w(s) \{ \int_0^s (B_0(u)/f(F^{-1}(u))) du \} ds ]$$

$$= \int_0^1 w(s) E(\int_0^s (B_0(u)/f(F^{-1}(u))) du) = 0$$

while

$$Var[\int_0^1 w(s) \{ \int_0^s (B_0(u)/f(F^{-1}(u))) du \} ds] = \int_0^1 \int_0^1 w(s)w(t)k(s,t) ds dt,$$

where k(s, t) is given by (2).

**Proof of Theorem 2.1.** Following Lemmas 2.2 and 2.3, it would then be sufficient to show that

$$\int_0^1 w(s) \{ \int_0^s (B_N(u)/f(F^{-1}(u))) du \} d(H_N(s) - s)$$

converges to zero in probability when  $N \to \infty$ ,  $n/N \to \rho$ ,  $0 < \rho < 1$ . The proof is analogous to that of Corollary 5.6.4, Csorgo and Revesz(1981), with their  $B_{N2}(y)$  replaced by our

$$\int_0^s (B_N(u)/f(F^{-1}(u))) du.$$

Thus, as  $N \rightarrow \infty$ , we have

$$\int_0^1 w(s) \{ \int_0^s (B_N(u)/f(F^{-1}(u))) du \} dH_N(s)$$

$$\xrightarrow{D} \int_0^1 w(s) \left\{ \int_0^s \left( B_0(u) / f(F^{-1}(u)) \right) du \right\} ds.$$

### 3. Variance Estimator

In order to make full use of Theorem 2.1 in problems of inference, one needs to estimate  $\sigma^2$ . In this context, the jackknifing turns out to be very useful. Let

$$L_n(p) = \int_0^1 J(u) F_{n-1}(u) du,$$

where

$$J(u) = I[0, p](u), 0 \le p \le 1.$$
(3)

and let  $L_{n-1}^{(i)}$  be the L-statistic based on  $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$  (i.e., on a sample size n-1 when  $X_i$  has been removed from the original sample). Then, the ordinary jackknife  $L_n(p)$  may be written as

$$L_n^* = n^{-1} \sum_{i=1}^n L_{n,i}$$
,

where

$$L_{n,i} = n L_n(p) - (n-1) L_{n-i}^{(i)}$$
 i=1, ..., n.

Also, the jackknife variance estimator of  $L_n(p)$  is defined by

$$S_J^2 = (n-1)^{-1} \sum_{l=1}^{n} (L_{n,l} - L_n^*)^2$$

Parr and Schucany (1982) proved that for  $L_n(p)$  with "smooth" J,

$$S_J^2 \to \sigma^2(J, F)$$

with probability one, where

$$\sigma^{2}(J, F) = \int_{0}^{1} \int_{0}^{1} J(u)J(v)\{\min(u, v) - uv\} dF^{-1}(u) dF^{-1}(v).$$

Since the derivative of J exists and is continuous on the unit interval except that the derivative of J fails to exist at the single point p, Theorem 2 of Parr and Schucany(1982) applies. Let

$$S_{Jp}S_{Jq} = (n-1)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}(L_{n,i}-L_{n}^{*})(L_{n,j}-L_{n}^{*}),$$

$$S_n^2 = \int_0^1 \int_0^1 w(p)w(q) S_{Jp}S_{Jq} dpdq,$$

and  $S_m^2$  be the corresponding quantity of  $S_n^2$  for the second sample of  $Y_j$ 's. Define

$$S_N^2 = \frac{1}{n} S_n^2 + \frac{1}{m} S_m^2.$$

We shall now establish the consistency of  $S_N^2$ , the estimator of  $\frac{N}{nm}\sigma^2$ , using the above result of Parr and Schucany (1982) and the continuous mapping theorem.

**Theorem 3.1.** Under  $H_0$ ,  $(nm/N)S_N^2 \rightarrow \sigma^2$  with probability one, where the

score function J is given by (3).

**Proof.** Since  $S_J^2 \to \sigma^2(J, F)$  with probability one, it is obvious that  $S_{Jp}S_{Jq}$  converges to

$$\int_0^p \int_0^q \{\min(u, v) - uv\} dF^{-1}(u) dF^{-1}(v)$$

with probability one. Then, by the continuity of the integration operator,

$$S_n^2 \to \int_0^1 \int_0^1 w(p)w(q) \int_0^p \int_0^q \{\min(u, v) - uv\} dF^{-1}(u) dF^{-1}(v) dpdq$$

$$S_m^2 \to \int_0^1 \int_0^1 w(p)w(q) \int_0^p \int_0^q \{\min(u, v) - uv\} d G^{-1}(u) d G^{-1}(v) dpdq$$

with probability one. Thus, under  $H_0$ , (nm/N)  $S_N^2 \to \sigma^2$  with probability one.

Therefore we can use this jackknife estimator of  $\sigma^2$  and Slutsky's theorem to conclude

$$T_N/(nm/N)^{1/2}S_N \to N(0,1)$$

the

which is a useful result that can be used in practice when dealing with large samples.

#### REFERENCES

Billingsley, P. (1968). Convergence of Probability Measures. John Wiley, New York.

Csorgo, M. and Revesz, P. (1981). Strong Approximations in Probability and Statistics. Academic Press, New York.

Deshpande, J.V. and Singh, P. (1985). Testing for second order stochastic dominance. *Commun.Statist.-Theor.Math.*, 14, 887-893.

Hadar, J. and Russell, W.R. (1969). Rules for ordering uncertain prospects. *Amer. Economic Rev.*, **59**, 25-34.

Hanoch, G. and Levy, H.(1969). The efficacy analysis of choices involving risk. *Rev. Econ. Studies*, **36**, 335-346.

Kaur, A., Rao, B.L.S.P. and Singh, H. (1994). Testing for second order stochastic dominance of two distributions. *Econometric Theory*, **10**, 849-866.

Loeve, M. (1963). *Probability Theory*. D. Van Norstrand Co., Inc, Princeton, N.J.

McFadden, D. (1989). Testing for stochastic dominance. In Studies in the Economics of Uncertainty: In Honor of Josef Hadar (eds T.B. Fomby and T.K. Seo). Springer Verlag, New York.

Parr, W.C. and Schucanny, W.R. (1982). Jackknifing L-statistics with smooth weight functions. *J. Amer. Stat. Assoc.*, 77, 692-638.

Whitmore, G.A. and Findlay, M.C. (1978). Stochastic Dominance: An Approach to Decision-Making Under Risk. Lexington Books, Toronto.

Whitt, W. (1980). Some useful functions for functional limit theorems. *Math.Op.Res.*, **5**, 67-85.